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Energy spectral element method for longitudinal acoustic propagation in ducts

Método dos elementos espectrais de energia para propagação acústica longitudinal em dutos

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Elson Cesar Moraes
ORCID: https://orcid.org/0000-0003-3717-5851
Mechanical Engineering Course, Federal University of Maranhão -UFMA, São Luís-MA, Brazil
E-mail: elson.cm@ufma.br
Vilson Souza Pereira
ORCID: https://orcid.org/0000-0003-0814-0449
Mechanical Engineering Course, Federal University of Maranhão-UFMA, São Luís-MA, Brazil
E-mail: vilson.sp@ufma.br
José Maria Campos Dos Santos
ORCID: https://orcid.org/0000-0002-3277-3492
Computational Mechanics Department, Faculty of Mechanical Engineer, State University of Campinas-UNICAMP, Campinas-SP, Brazil
E-mail: zema@fem.unicamp.br

ABSTRACT

Sound field behavior in an acoustic enclosure is an important part of the design of transportation vehicle passenger cabin, concert halls, conference rooms, and etc. Different analysis methods are available and have strengths and weaknesses. Low frequency band envelopes and negligible absorption in the walls can be modeled by Modal Analysis or Finite Element Method. However, as the frequency band increases, both methods become computationally intensive and Statistical Energy Analysis or the Sabine model can be an efficient approach. However, these methods do not take into account any spatial variation within the enclosure. The Energy Flow Analysis (EFA) solution for acoustic enclosures can be done analytically. In this paper, the Energy Spectral Element Method (ESEM) is formulated and applied to predict the spatial distribution of energy flux and density of acoustic ducts at high frequencies. ESEM is a matrix methodology based on EFA to solve acoustic and structural vibration problems. In this work, numerical models involving simple and coupled one-dimensional acoustic ducts are generated by ESEM, and the results are compared with energy densities calculated from the pressure fields predicted by the Spectral Element Method (SEM).

Keywords: Energy Flow Analysis; Waveguides; Energy Spectral Element Method; Acoustic Enclosure.

RESUMO

O comportamento do campo sonoro em um gabinete acústico é uma parte importante do projeto de cabines de passageiros de veículos de transporte, salas de concerto, salas de conferência e etc. Diferentes métodos de análise estão disponíveis e têm pontos fortes e fracos. Gabinetes submetidos a banda de baixa frequência e absorção desprezível nas paredes podem ser modelados por Análise Modal ou FEM. No entanto, à medida que a banda de frequência aumenta, ambos os métodos se tornam intensivos e a Análise Estatística de

Energia ou o modelo de Sabine podem ser eficientes. Esses métodos não levam em conta qualquer variação espacial dentro do recinto. A solução de Análise de Fluxo de Energia (EFA) para caixas acústicas pode ser feita analiticamente. Neste artigo, o Método dos Elementos Espectrais de Energia (ESEM) é aplicado para prever a distribuição espacial da densidade e fluxo de energia de dutos acústicos em altas frequências. ESEM é baseado em EFA para resolver problemas de vibração acústica e estrutural. Modelos numéricos para dutos acústicos unidimensionais simples e acoplados são gerados pelo ESEM, os resultados são comparados com densidades de energia calculadas pelo Método dos Elementos Espectrais (SEM).

Palavras-chave: Análise de Fluxo de Energia; Guias de onda; Método dos Elementos Espectrais de Energia; Gabinete Acústico.

INTRODUCTION

Currently, for product designs such as airplane, automobile, household appliance, lecture rooms, and etc., the sound level is a very important factor to determine the products acceptance by consumers. Despite of several types of tools available to design engineers including, Sabine room acoustic model, Modal Analysis, Statistical Energy Analysis (SEA), Finite Element Method (FEM), and Energy methods, scientists and researchers have been developing new predictive tools. FEM have the inherent characteristic to generate high order computational models, which makes their use inadequate at high modal density range. Therefore, complex structure behavior at high-frequency band is still an active research subject. A commonly used high frequency modeling approach is SEA (Lyon and Dejong, 1995). Its limitation comes from the inability to calculate the energy spatial variation in each subsystem. EFA is an enhanced SEA, since it provides the spatial energy distribution within thesubsystems (Wohlever and Bernhard, 1992). The ESEM consists of applying the same matrix methodology of FEM to the analytical solution of EFA (Santos *at al.*, 2008). In this paper an extension of ESEM formulation for acoustic wave propagation problem in a single and coupled finite one-dimensional waveguide (duct) is presented. The model is based on the assumption of plane waves and a gas loss factor is included. Single and coupling circular cross section duct examples are simulated by ESEM and energy density and flow results are presented. Results for coupled ducts examples with discontinuities due to cross section area and gas property variation are shown too. These results are verified with ones calculated by the spectral element (Donadon and Arruda, 2003).

BASIC THEORY

One-dimensional Acoustic Energy Spectral Element

An extension of Energy Spectral Element Method (ESEM) to the acoustic medium is proposed, which allows solving the approximated energy flow solution by applying the same matrix scheme as FEM and SEM. For steady state condition, harmonic excitation, small gas loss factor (η << 1), the time and space averaged energy density for acoustic plane waves in a one-dimensional waveguides can be written as (Wohlever and Bernhard, 1992),

$$-\frac{c^2}{\eta\omega}\nabla^2\langle\bar{e}\rangle + \eta\omega\langle\bar{e}\rangle = \Pi \tag{1}$$

where $\langle\ \rangle$ represents time-average, $\bar{}$ represents space-average, e is the energy density, η is the gas loss factor, ω is the circular frequency, c is the sound velocity and Π is the input power. The energy flow (intensity) is related to the energy density by:

$$\langle\bar{q}\rangle = -\frac{c}{\eta\omega}\nabla\langle\bar{e}\rangle \tag{2}$$

The one-dimensional homogenous solution of Eq. (1) is given by:

$$\langle \bar{e} \rangle (x) = G e^{\eta k x} + H e^{-\eta k x} \tag{3}$$

where $k = \omega / c$ is the wavenumber, G and H are constant coefficients determined from boundary conditions. By applying end conditions in a two-node acoustic energy spectral element (Fig. 1a), the energy density at any arbitrary point along the element is obtained as:

$$\langle \bar{e} \rangle (x) = \underbrace{\left(\frac{e^{\eta k x} - e^{\eta k (2L - x)}}{1 + e^{2\eta k L}} \right)}_{\mathrm{h}_1(x)} \langle \bar{e}_1 \rangle + \underbrace{\left(\frac{e^{\eta k (L - x)} - e^{\eta k (L + x)}}{1 + e^{2\eta k L}} \right)}_{\mathrm{h}_2(x)} \langle \bar{e}_2 \rangle \tag{4}$$

Figure 1–Two-node one-dimensional acoustic waveguide element by: (a) ESEM; (b) SEM

(a) (b)

Source: Authors

where $\mathrm{h}_1(x)$ and $\mathrm{h}_2(x)$ are the interpolation functions of the energy spectral element. By substituting Eq. (4) in Eq. (2) the energy flow at any arbitrary point along the element can be written as,

$$\langle \bar{q} \rangle (x) = -\frac{k c^2}{\omega (1 - e^{2\eta k L})} \left[\left(e^{\eta k x} - e^{\eta k (2L - x)} \right) \langle \bar{e}_1 \rangle + \left(e^{\eta k (L + x)} + e^{\eta k (L - x)} \right) \langle \bar{e}_2 \rangle \right] \tag{5}$$

By applying end conditions in a two-node acoustic spectral energy element (Fig. 1a) the energy flow can be written in a matrix form as:

$$\begin{Bmatrix} \langle \bar{q}_1 \rangle \\ \langle \bar{q}_2 \rangle \end{Bmatrix} = \underbrace{-\frac{k c^2}{\omega (1 - e^{2\eta k L})} \begin{bmatrix} 1 + e^{2\eta k L} & -2 e^{\eta k L} \\ -2 e^{\eta k L} & 1 + e^{2\eta k L} \end{bmatrix}}_{\boldsymbol{K_E}} \begin{Bmatrix} \langle \bar{e}_1 \rangle \\ \langle \bar{e}_2 \rangle \end{Bmatrix} \tag{6}$$

where, $\mathbf{K}_E$ is the spectral energy flow element matrix.

Reactive-type exhaust mufflers use the ability of a cross section area change to attenuate the sound energy transmitted in a duct. Also, the medium property change in a duct will attenuate the sound energy transmitted. In the acoustic filter theory, the medium is assumed to be stationary and the wave propagation is governed by the 1-D wave equation. To account for these discontinuities in the energy model, additional coupling relationships need to be formulated and inserted at these connection points. Since a rod and an acoustic duct are ruled by the same wave equation, the joint element used in this study is the same proposed by Cho and Bernhard (1998) for structural coupling type rod–rod. Then, the acoustic coupling relationship is obtained as:

$$\begin{Bmatrix} \langle \bar{q}_i \rangle \\ \langle \bar{q}_j \rangle \end{Bmatrix} = \underbrace{-\frac{\tau_{ij}}{2r_{ii}} \begin{bmatrix} c_i & -c_j \\ -c_i & c_j \end{bmatrix}}_{J} \begin{Bmatrix} \langle \bar{e}_i \rangle \\ \langle \bar{e}_j \rangle \end{Bmatrix} \tag{7}$$

where τ_{ij} and r_{ii} are the power transmission and reflection coefficients at the coincident nodes between elements *i* and *j*.

One-dimensional Acoustic Spectral Element

Considering that SEM will be used to verify the proposed element, a brief formulation of them is presented here. Although proposed by Doyle (1997) to study structural wave propagation, the governing equations for acoustic and structural one-dimensional waveguides are the same. The Helmholtz equation is the linearized, lossless wave equation for sound propagation in fluids, re-written here as a lossy wave equation as (Kinsler et al, 1982):

$$\frac{\partial^2 \hat{p}}{\partial x^2} - {k_c}^2 \hat{p} = 0 \tag{8}$$

where ^ indicates frequency domain function, and *p* is the acoustic pressure. The complex wavenumber is included to account for the energy absorption mechanism in the gas, given by $k_c \approx k(1 - i\eta/2)$, where $i = \sqrt{-1}$. The linear Euler equation states the relationship between particle velocity and acoustic pressure by:

$$\hat{u} = -\frac{1}{i\omega\rho}\frac{\partial \hat{p}}{\partial x} \tag{9}$$

where ρ is the mass density. The general solution of Eq. (9) can be written as,

$$\hat{p}(x) = Ae^{-ik_c x} + Be^{ik_c x} \tag{10}$$

where *A* and *B* are constant coefficients determined from the boundary conditions. Applying end conditions in a two-node acoustic spectral element (Fig. 1b), the acoustic pressure at any arbitrary point along the element is,

$$\hat{p}(x) = \underbrace{\left(\frac{e^{ik_c x} - e^{ik_c(2L-x)}}{1 - e^{2ik_c L}}\right)}_{\hat{g}_1(x)} \hat{p}_1 - \underbrace{\left(\frac{e^{ik_c(L-x)} - e^{ik_c(L+x)}}{1 - e^{2ik_c L}}\right)}_{\hat{g}_2(x)} \hat{p}_2 \tag{11}$$

where, $g_1(x)$ and $g_2(x)$ are the interpolation functions of the spectral element. By substituting Eq. (11) in Eq. (9) the particle velocity at any arbitrary point along the element can be written as,

$$\hat{u}(x) = -\frac{k_c}{\omega\rho(1-e^{2ik_cL})}\left[\left(-e^{ik_cx} - e^{ik_c(2L-x)}\right)\hat{p}_1 + \left(e^{ik_c(L+x)} + e^{ik_c(L-x)}\right)\hat{p}_2\right] \quad (12)$$

Applying end conditions in the two-node acoustic spectral element (Fig. 1b), the particle velocity can be written in a matrix form as:

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix} = \underbrace{-\frac{k_c}{\omega\rho(1-e^{2ik_cL})}\begin{bmatrix} 1+e^{2ik_cL} & -2e^{ik_cL} \\ -2e^{ik_cL} & 1+e^{2ik_cL} \end{bmatrix}}_{\boldsymbol{K_S}}\begin{Bmatrix} \hat{p}_1 \\ \hat{p}_2 \end{Bmatrix} \quad (13)$$

where, $\mathbf{K}_S$ is the dynamic spectral element matrix.

The time-average energy density in an acoustic medium is the sum of potential and kinetic energy densities as:

$$\langle e \rangle = \frac{1}{4}\left(\rho\hat{u}\hat{u}^* + \frac{1}{\rho c^2}\hat{p}\hat{p}^*\right) \quad (14)$$

where * represents complex conjugate. Time-average energy flow (intensity) in an acoustic medium is written as:

$$\langle q \rangle = \frac{1}{2}\Re\{\hat{p}\hat{u}^*\} \quad (15)$$

where $\Re$ is the real part of a complex number.

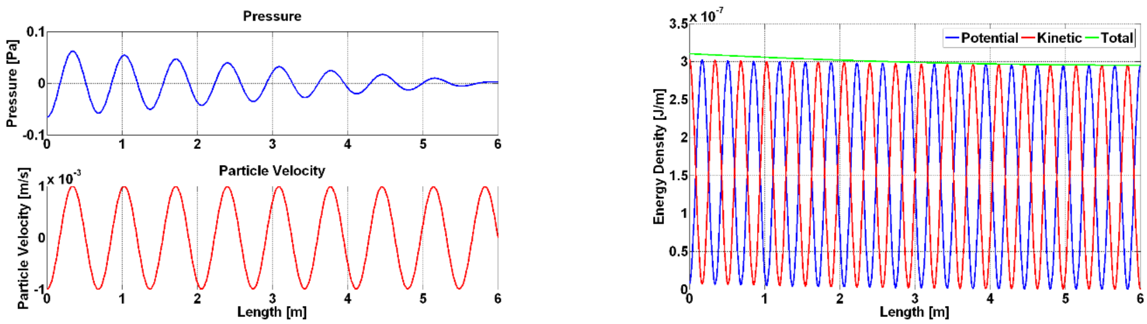
SIMULATED RESULTS

Single Element

The one-dimensional acoustic waveguide consists of a rigid wall cylindrical duct with an unflanged open end and harmonically excited at the other end. The acoustic medium is the air at 20ºC (ρ = 1.21 kg/m^3; c = 343 m/s; and η = 2.86×10^{-3}), and the duct geometry is R = 2.0×10^{-3} m and L = 6.0m. The harmonic excitation for the SEM model is the particle velocity with a magnitude u = 1×10^{-3} m/s, while for the ESEM model it is the corresponding input power calculated with the Eq.(15). To account for the duct unflanged open end, a radiation impedance is included in the SEM model as $Z_{mL} = \rho cS\,(0.25k^2R^2 - i0.6kR)$, while a corresponding energy flow is incorporated in the ESEM model as $\langle q \rangle = \alpha c\langle e \rangle$, where $\alpha = \tau/(2-\tau)$ and $\tau = 1-|(Z_{mL}-\rho c)/(Z_{mL}+\rho c)|^2$.

Figure 2a shows the responses in acoustic pressure and particle velocity for one-dimensional acoustic SEM element at the frequency f = 500 Hz. A typical plot of the time-averaged total, potential and kinetic energy densities in the SEM element is shown in Fig. 2b. The harmonic portions of the potential and kinetic energy density sum to give what appears to be a constant value of energy density along the element. Although it is not so evident in Fig. 2b, there is a slight exponential decay of the total energy density distribution due to the dissipation of energy caused by the gas loss factor. Figure 3 presents the energy density and energy flow comparison calculated by ESEM and SEM in a frequency f = 500 Hz. Energy density and flow calculated by ESEM and SEM match exactly. Although not shown here this comparison was made to other frequency values and the results were the same.

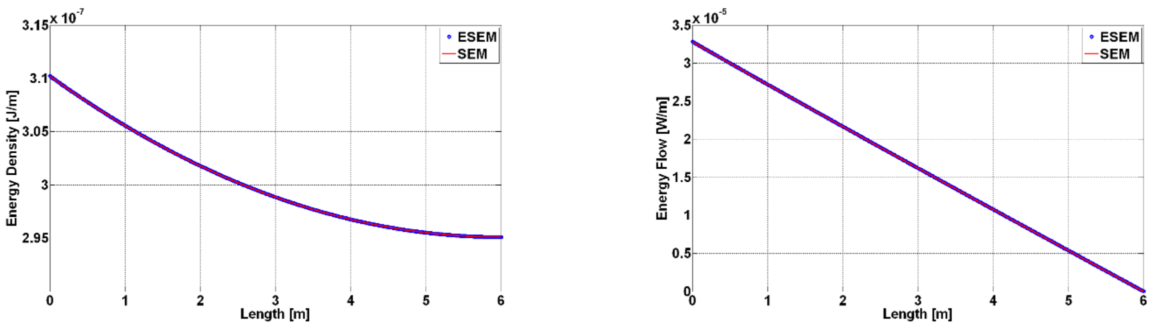
Figure 2 – One-dimensional acoustic waveguide SEM element responses at frequency f = 500Hz: (a) acoustic pressure and particle velocity; (b) time-average potential, kinetic and total energy density



(a) (b)

Source: Authors

Figure 3 – Comparison of ESEM and SEM one-dimensional acoustic waveguide element responses at frequency f=500Hz: (a) energy density; (b) energy flow



(a) (b)

Source: Authors

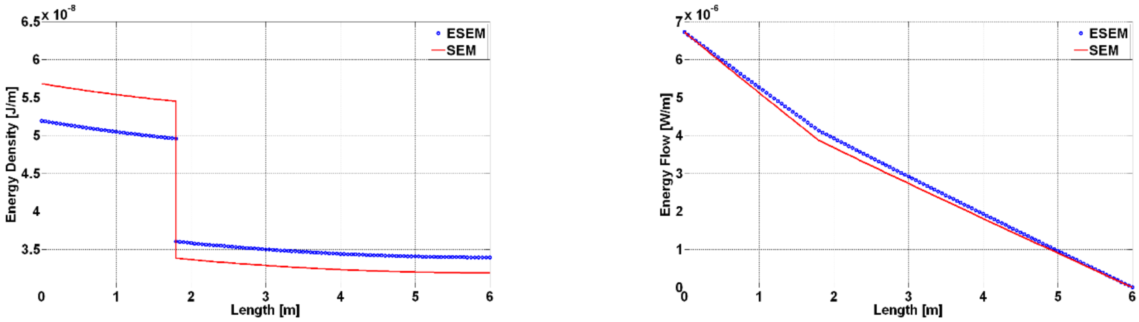
Coupled Elements

The coupled acoustic system consists of two one-dimensional acoustic waveguide (rigid wall ducts) with different geometry (cross section radius and length) or medium property (sound velocity, gas density and loss factor) connected each other. The system is harmonically excited at the left end and opened in the right end (unflanged). For the geometric discontinuity the duct dimensions for element 1 are $R_1 = 2.0\times10^{-3}$ m and L_1 = 1.8 m, while for element 2 are $R_2 = 4.0\times10^{-3}$ m and L_2 = 4.2 m. Both elements contains air at 0ºC (ρ = 1.21 kg/m^3; c = 343.0 m/s; and η = 5.8×10^{-3}) as acoustic medium. For the medium discontinuity the element 1 contains air at 0ºC (ρ_1

= 1.293 kg/m^3, c_1 = 331.6 m/s, η_1 = 5.8×10^{-3}), while the element 2 contains hydrogen at 0ºC (ρ_2 = 0.09 kg/m^3; c_2 = 1269.5 m/s; and η_2 = 8.8×10^{-3}). Both elements have same geometric dimensions (R = 5.0×10^{-3} m and L = 6 m). The excitation and end conditions are the same as the single element problem.

Figure 4 shows the frequency-averaged energy density and flow responses, calculated by ESEM and SEM, for two coupled elements with cross section area discontinuity in 1/3-octave frequency band with center frequency f_c = 0.8 kHz. Figure 5 shows the same for f_c = 8.0 kHz. Both methods present a typical plot of frequency-averaged energy density (Figs. 4a and 5a), which includes an energy decaying along the element length with a sudden step at the position of the cross section area change. Whereas, the energy flow plots (Figs. 4b and 5b) show a similar behavior without the step.

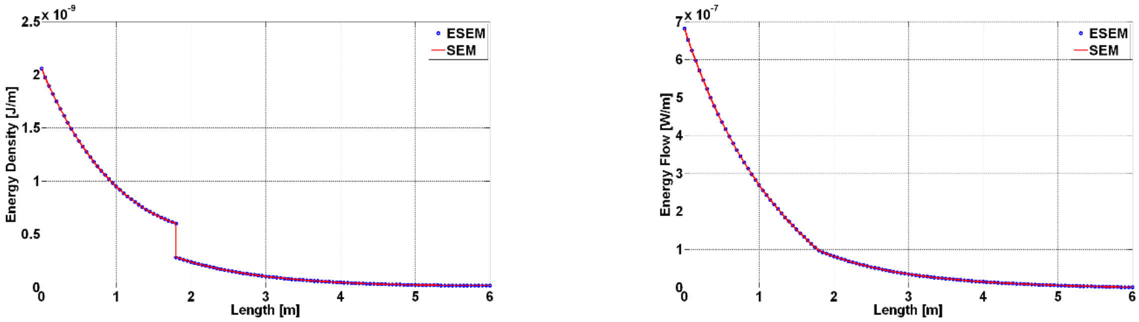
Figure 4 – Cross section discontinuity by ESEM and SEM with f_c = 0.8 kHz: (a) Energy density; (b) Energy flow



(a) (b)

Source: Authors

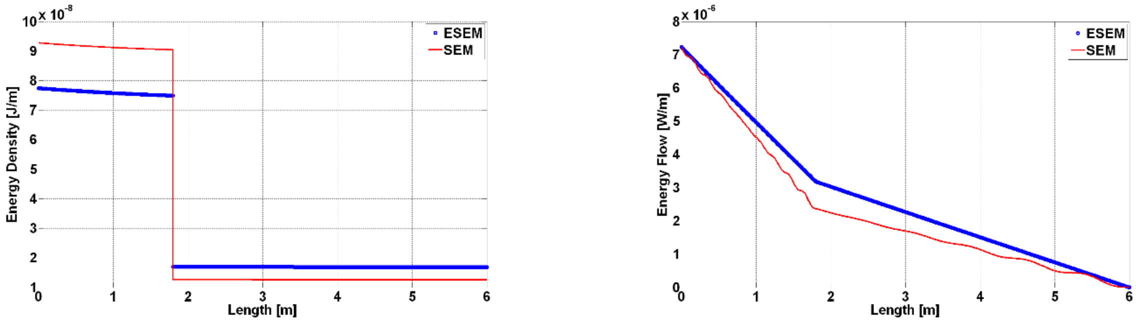
Figure 5 – Cross section discontinuity by ESEM and SEM with fc = 8.0 kHz: (a) Energy density; (b) Energy flow



(a) (b)

Source: Authors

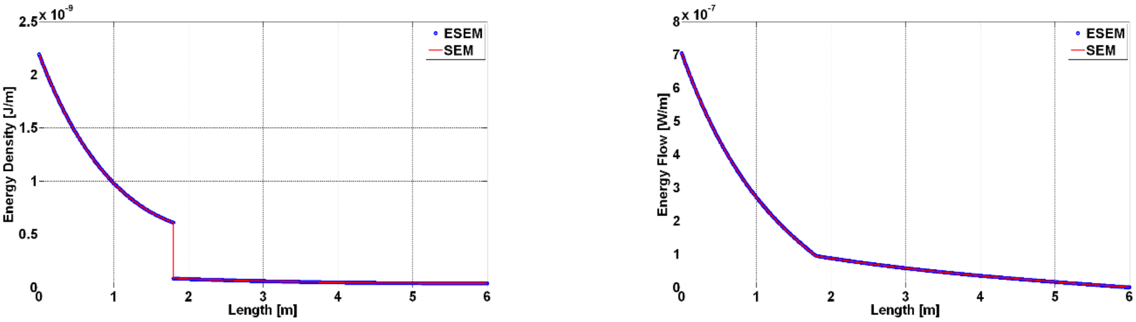
Figure 6 – Medium property discontinuity by ESEM and SEM with fc = 0.8 kHz: (a) Energy density; (b) Energy flow



(a) (b)

Source: Authors

Figure 7 – Medium property discontinuity by ESEM and SEM with fc = 8.0 kHz: (a) Energy density; (b) Energy flow



(a) (b)

Source: Authors

Nevertheless, a comparison between frequency-averaged energy density and flow responses by ESEM and SEM shows a mismatch at low frequency band (f_c = 0.8 kHz), and a perfect agreement at high frequency band (f_c = 8.0 kHz). Figure 6 and 7 show similar results for the two coupled elements with medium property discontinuity. Comparisons were made at different frequency bands, and the results are similar.

FINAL REMARKS

In this work, the thermal analogy proposed by Wohlever and Bernhard (1992) to model mechanical energy flow in structural systems is investigated for acoustic systems. Energy density and flow were derived from the classical lossy Helmholtz equation solution for harmonically excited duct. An extension of energy spectral element method to the acoustic one-dimensional waveguides is proposed. Predictions made with ESEM for one-dimensional acoustic waveguides are verified using an exact solution of the wave equation obtained by the spectral element method. Some examples are simulated and results obtained by ESEM and SEM are compared and discussed. The configurations treated consist of acoustic ducts composed of single element and two coupled elements. The main divergences between the energy density and energy flow results obtained with SEM and ESEM stem from the validity limits of ESEM formulation and coupling relationships in the frequency band of interest. ESEM reaffirms to be suitable for high frequencies, and it produces good results when the analyses are performed inside the validity region for the method.

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